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Conformally Flat Space-Time of Spherical Symmetry in Isotropic Coordinates

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Abstract

When we study spherically symmetric distributions of non-charged, perfect fluid in general relativity, a general expression giving the metric can be obtained under the assumption that the space-time is conformally flat. Formulae for the metric, matter density and pressure are given in isotropic coordinates.

Conformally flat space-times constitute an especially important class of Riemannian space-times; it is sufficient to mention that the space-times of relativistic cosmology belong here. It is also a well-known feature of the internal Schwarzschild solution (Schwarzschild, 1916) that its metric is conformally flat. It may be easily demonstrated that this is the only static and spherically symmetric solution of the Einstein equations that is conformally flat. Now it could be asked with respect to the non-static solutions of Einstein's equations in the case of spherical symmetry whether it is possible to obtain exact expressions for extensions of the internal Schwarzschild solution.

A search for non-static solutions of Einstein's equations for a perfect fluid will be made in the isotropic coordinates in which the metric has the form

$$ds^2 = -e^{\lambda}(dr^2 + r^2\, d\vartheta^2 + r^2 \sin^2\vartheta\, d\varphi^2) + e^{\nu}\, dt^2 \tag{1}$$

The advantage of the isotropic coordinate system over other ones is due to the fact that the line element of any spherically symmetric space-time can be brought into the form (1). The detailed form of Einstein's field equations

$$R_\nu{}^\mu - \tfrac{1}{2}R\delta_\nu{}^\mu = -8\pi T_\nu{}^\mu \tag{2}$$

with the metric (1) is given in Section 3 of a previous paper (Kuchowicz, 1971). The non-vanishing components of the energy-momentum tensor are here: $T_1^1 = T_2^2 = T_3^3 = -p$, $T_4^4 = \rho$, where ρ is the energy density, and p pressure, as measured by a local comoving observer. Now, the condition of pressure isotropy ($T_1^1 = T_2^2$) leads to the following relation:

$$\nu'' + \lambda'' + \tfrac{1}{2}(\nu'^2 - \lambda'^2) - \nu'\lambda' - \frac{\lambda' + \nu'}{r} = 0 \tag{3}$$

Here dots denote differentiation with respect to t, and primes with respect to r.

Another relation follows from the condition $T_4^1 = 0$:

$$\dot{\lambda}' - \tfrac{1}{2}\dot{\lambda}\nu' = 0 \tag{4}$$

It was shown in the previous paper (Kuchowicz, 1971) that with a solution of equation (4),

$$e^{\nu/2} = \dot{\lambda}F(t) \tag{5}$$

we obtain from equation (3) a partial differential equation for the function λ only.

Use will now be made from the condition of conformal flatness of our space-time. The conformal curvature tensor which is defined as

$$C_{\lambda\mu\nu\sigma} = R_{\lambda\mu\nu\sigma} - \tfrac{1}{2}(g_{\lambda\sigma}R_{\mu\nu} + g_{\mu\nu}R_{\lambda\sigma} - g_{\lambda\nu}R_{\mu\sigma} - g_{\mu\sigma}R_{\lambda\nu}) + \tfrac{1}{6}R(g_{\lambda\sigma}g_{\mu\nu} - g_{\lambda\nu}g_{\mu\sigma}) \tag{6}$$

should be equal to zero. Since in spherically symmetric space-times it has essentially one independent component only, our condition leads to one relation only:

$$\nu'' - \lambda'' + \tfrac{1}{2}(\nu' - \lambda')^2 + \frac{\lambda' - \nu'}{r} = 0 \tag{7}$$

This gives, after one integration,

$$\lambda' - \nu' = \frac{4r}{F_1(t) - r^2} \tag{8}$$

where $F_1(t)$ is an arbitrary function of t. When we insert this, and equation (5) into equation (3), we have

$$\lambda'' - \tfrac{1}{2}(\lambda')^2 - \frac{\lambda'}{r} = 0 \tag{9}$$

The latter equation has the same form as in the static case (when the internal Schwarzschild solution is obtained), only the integration constants now become functions of t. Finally we obtain

$$e^{\lambda} = \frac{F_3(t)}{[F_2(t) - r^2]^2}, \qquad e^{\nu} = F_4(t)\left(\frac{F_1(t) - r^2}{F_2(t) - r^2}\right)^2 \tag{10}$$

This metric with the four arbitrary functions $F_i(t)$ is the most general conformally flat metric of spherical symmetry in isotropic coordinates. The corresponding expressions for energy density and pressure are

$$8\pi\rho = -\frac{12F_2}{F_3} + \frac{3}{4F_4}\left(\frac{F_2 - r^2}{F_1 - r^2}\right)^2\left[\frac{\dot{F}_3}{F_3} - 2\frac{\dot{F}_2}{F_2 - r^2}\right]^2$$

$$\begin{aligned}8\pi p = {} & \frac{4}{F_3(F_1 - r^2)}\{2F_1F_2 - F_2^2 + r^2(F_1 - 2F_2)\} \\ & + \frac{1}{F_4}\left(\frac{F_2 - r^2}{F_1 - r^2}\right)^2 \cdot \left\{-\frac{\ddot{F}_3}{F_3} + \frac{3}{4}\left(\frac{\dot{F}_3}{F_3}\right)^2\right. \\ & + \frac{\dot{F}_3}{F_3}\frac{\dot{F}_2}{F_2 - r^2} - 3\left(\frac{\dot{F}_2}{F_2 - r^2}\right)^2 + 2\frac{\ddot{F}_2}{F_2 - r^2} \\ & \left. + \frac{1}{2}\left(\frac{2\dot{F}_1}{F_1 - r^2} + \frac{\dot{F}_4}{F_4}\right)\left(\frac{\dot{F}_3}{F_3} - \frac{2\dot{F}_2}{F_2 - r^2}\right)\right\}\end{aligned} \tag{11}$$

From the expressions for energy density and pressure it is easy to see that when the functions F_i do not depend on time, our solution reduces to the internal Schwarzschild solution. The ~~class of~~ Robertson-Walker metrics may be obtained from our expressions (10) with the substitution $F_1 = F_2 =$ constant. The metric we derived may be thus used both for a description of general relativistic non-static spheres, along the lines presented in some previous papers (e.g. Kuchowicz, 1971), or for cosmological models.

It seems to be worthwhile to mention that we have solved equation (7) provided $\lambda' \neq \nu'$. If this condition is not fulfilled then the general spherically symmetric metric has the form

$$ds^2 = e^{\sigma}[dt^2 - dr^2 - r^2\,d\vartheta^2 - r^2\sin^2\vartheta\,d\varphi^2] \tag{12}$$

where the function σ is the solution of the two equations

$$2\sigma'' - (\sigma')^2 - \frac{2\sigma'}{r} = 0 \tag{13}$$

$$\dot{\sigma}' - \tfrac{1}{2}\dot{\sigma}\sigma' = 0 \tag{14}$$

We have

$$e^{\sigma} = \frac{G_1(t)}{[r^2 + G_2(t)]^2} \tag{15}$$

with two arbitrary functions of time, $G_1(t)$ and $G_2(t)$. Energy density and pressure are now if G_1 is constant

$$\begin{aligned}8\pi\rho &= G_1^{-1}[12G_2 - 3\dot{G}_2^2] \\ 8\pi p &= G_1^{-1}[(4 + 2\ddot{G}_2)r^2 + 2G_2\ddot{G}_2 - 3\dot{G}_2^2 - 8G_2]\end{aligned} \tag{16}$$

We see that pressure does not depend on t provided we have

$$G_2(t) = -t^2 + A_1 t + A_2 \tag{17}$$

and A_i are constants.

The solution (15) may be obtained from equation (10) by the formal substitution $F_4(F_1)^2 = F_3$, $F_4 = 0$. Due to the fact that it is possible for any spherically symmetric space-time to apply the isotropic coordinates, it is evident that our formulae give the general conformally flat metric of spherical symmetry; in some cases this may be brought into another form that would be consistent with other coordinate systems, e.g. the canonical ones.

References

Kuchowicz, B. (1971). *Acta Physica Polonica*, **B2**, 657; here references to numerous other papers may be found.

Schwarzschild, K. (1916). *Sitzungsberichte der Preussischen Akademie der Wissenschaften zu Berlin*, 424.